Polynomials with only real zeros

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Abstract

Conditions which ensure that a combination of real polynomials, which have real interlacing zeros, continues to have only real zeros are derived. This gives a generalization of a result of Haglund and is proved using a unified method of Liu-Wang-Yeh.

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1 Introduction

Let

\[ RZ = \{ P(x) \in \mathbb{R}[x]; P(x) \text{ has only real zeros} \} \]

\[ PF = \{ P(x) \in RZ; \text{ all coefficients of } P(x) \in \mathbb{R}_{\geq 0} \}. \]

Let \( f, g \in RZ \) and let \( \{ r_i \} \) and \( \{ s_j \} \) be all respective zeros of \( f \) and \( g \) in non-increasing order. Following Wagner [6], we say that \( g \) alternates \( f \) if \( \deg f = \deg g = n \) and

\[ s_n \leq r_n \leq s_{n-1} \leq \ldots \leq s_2 \leq r_2 \leq s_1 \leq r_1; \tag{1.1} \]

we say that \( g \) interlaces \( f \) if \( \deg f = \deg g + 1 = n \) and

\[ r_n \leq s_{n-1} \leq \ldots \leq s_2 \leq r_2 \leq s_1 \leq r_1. \tag{1.2} \]

The notation \( g \preceq f \) denotes either \( g \) alternates \( f \) or \( g \) interlaces \( f \). If no equality sign occurs in (1.1) (respectively (1.2)), then we say that \( g \) strictly alternates \( f \) (respectively \( g \) strictly interlaces \( f \)). Let \( g \prec f \) denote either \( g \) strictly alternates \( f \) or \( g \) strictly interlaces \( f \).

Polynomials with only real zeros arise often in combinatorics and other branches of mathematics (see [1], [4]). Let \( a_0, a_1, \ldots \) be a sequence of nonnegative real numbers. It is unimodal if

\[ a_0 \leq a_1 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \text{ for some } k. \]

It is log-concave (LC), if

\[ a_i a_{i+1} \leq a_{2i} \text{ for all } i > 0. \]

Log-concavity implies unimodality. Unimodal and log-concave sequences occur naturally in combinatorics, algebra, analysis, geometry, computer science, probability and statistics.

One classical approach to unimodality and log-concavity of a finite sequence is to use Newton’s inequality: if the polynomial \( \sum_{i=0}^{n} a_i x^i \) with positive coefficients has only real zeros, then

\[ a_i^2 \geq a_{i-1}a_{i+1}(1 + \frac{1}{i})(1 + \frac{1}{n-i}) \]

for \( 1 \leq i \leq n-1 \), and the sequence \( a_0, a_1, \ldots, a_n \) is therefore unimodal and log-concave. Such a sequence of positive numbers whose generating function has only real zeros is called a Pólya-frequency (or PF) sequence. A deeper results is the following theorem which provides the basic link between finite Pólya-frequency sequence and polynomials having only real zeros.

**Aissen-Schoenberg-Whitney Theorem** ([5]).

A finite sequence \( a_0, \ldots, a_n \) of nonnegative number is Pólya-frequency sequence if and only if its generating function \( \sum_{i=0}^{n} a_i x^i \) has only real zeros.

In 2005, Wang and Yeh, [5], proved the following results.
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Theorem 1.1. (\[3, Theorem 1\]) Let \( f \) and \( g \) be real polynomials whose leading coefficients have the same sign. Suppose that \( f, g \in RZ \) and \( g \preceq f \). If \( ad \leq bc \), then \((ax + b)f(x) + (cx + d)g(x) \in RZ\).

Corollary 1.2. (\[3, Corollary 1\]) Suppose that \( f, g \in PF \) and \( g \) interlaces \( f \). If \( ad \geq bc \), then \((ax + b)f(x) + x(cx + d)g(x) \in RZ\).

In 2007, Liu and Wang, \[3\], gave the following sufficient conditions for a sequence of polynomials to have only real zeros based on the method of interlacing zeros:

Theorem 1.3. (\[3, Theorem 2.1\]) Let \( F, f, g \) be three real polynomials satisfying the following conditions;
(a) \( F(x) = a(x)f(x) + b(x)g(x) \) where \( a(x), b(x) \) are two real polynomials, such that \( \deg F = \deg f \) or \( \deg f + 1 \),
(b) \( f, g \in RZ \) and \( g \preceq f \),
(c) \( F \) and \( g \) have leading coefficients of the same sign,
(d) \( \forall r \in \mathbb{R}, f(r) = 0 \Rightarrow b(r) \leq 0 \).
Then \( F \in RZ \) and \( f \preceq F \). In particular, if \( g \prec f \) and \( b(r) < 0 \) whenever \( f(r) = 0 \), then \( f \prec F \).

Corollary 1.4. (\[3, Corollary 2.2\]) Let \( f \) and \( g \) be two real polynomials with positive leading coefficients \( \alpha \) and \( \beta \) respectively. Suppose that the following conditions are satisfied;
(a) \( f, g \in RZ \) and \( g \) interlaces \( f \),
(b) \( F(x) = (ax + b)f(x) + x(cx + d)g(x) \) where \( a, b, d \in \mathbb{R} \) with \( d \geq 0, d \geq b/a \) and either \( a > 0 \) or \( a < -\beta/\alpha \),
(c) all zeros of \( f \) are nonpositive if \( a > 0 \) and nonnegative if \( a < -\beta/\alpha \).
Then \( F \in RZ \). In addition, if each zero \( r \) of \( f \) satisfies \( -d \leq r \leq 0 \), then \( f \) interlaces \( F \).

Haglund in \[2\] used Corollary 1.4 to prove facts about rook polynomials in graph theory. Here we prove a generalization of Corollary 1.4 and give an example.

2 Result

Our main result is

Theorem 2.1. Let \( f \) and \( g \) be two real polynomials with both positive or both negative leading coefficients \( \alpha \) and \( \beta \) respectively. Suppose that the following conditions are satisfied;
(a) \( f, g \in RZ \) and \( g \) interlaces \( f \),
(b) \( F(x) = (ax + b)f(x) + x(cx + d)g(x) \) where \( a, b, c, d \in \mathbb{R} \) with \( a \neq 0 \) and \( d \geq bc/a \),

(c) if \( a > 0 \), then all zeros of \( f \) are nonpositive,

(d) if \( a < 0 \), then all zeros of \( f \) are nonnegative.

Then \( F \in \text{RZ} \). In addition, if \( c > 0 \) and \( -d/c \leq r \leq 0 \) for each zero \( r \) of \( f \), then \( f \) interlaces \( F \). If \( c = 0 \) and \( r \leq 0 \) for each zero \( r \) of \( f \), then \( f \) interlaces \( F \).

**Proof.** Let \( n \in \mathbb{N} \),

\[
f(x) := \alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{R}[x],
\]

and \( \alpha = \alpha_n \). Since \( g \) interlaces \( f \), then \( \deg g = n - 1 \). We distinguish two possibilities.

First, assume \( \alpha \) and \( \beta \) are positive.

If \( a > 0 \), by (c) we have \( r \leq 0 \) for each zero of \( f \). Thus, all coefficients of \( f \) are nonnegative, i.e., \( f \in \text{PF} \). Since \( g \) interlaces \( f \), we have \( g \in \text{PF} \). By Corollary \[1.2\] and \( ad \geq bc \), we deduce that \( F \in \text{RZ} \).

If \( a < 0 \), by (d) we have \( r \geq 0 \) for each zero of \( f \). By (a), all zeros of \( g \) are nonnegative. Thus, all coefficients of \( f \) and \( g \) are alternating in sign. Since \( \alpha \) is positive and all coefficients of \( f \) are alternating in sign, we see that \((-1)^i \alpha_{n-i} \geq 0 \) \((0 \leq i \leq n)\). Let

\[
f_1(x) := (-1)^n f(-x), \quad g_1(x) := (-1)^{n-1} g(-x) \quad \text{and} \quad F_1(x) := (-1)^n F(-x).
\]

We have

\[
f_1(x) = (-1)^n f(-x) = (-1)^n [\alpha_n(-x)^n + \alpha_{n-1}(-x)^{n-1} + \cdots + \alpha_1(-x) + \alpha_0]
= \alpha_n x^n + (-1)\alpha_{n-1} x^{n-1} + \cdots + (-1)^{n-1} \alpha_1 x + (-1)^n \alpha_0,
\]

and so all coefficients of \( f_1(x) \) are nonnegative, i.e., \( f_1 \in \text{PF} \). Similarly, \( g_1 \in \text{PF} \). Since \( g \) interlaces \( f \), the polynomial \( g_1 \) also interlaces \( f_1 \). From

\[
F_1(x) = (-1)^n F(-x) = (-1)^n [(a(-x) + b)f(-x) - x(c(-x) + d)g(-x)]
= (-ax + b)(-1)^n f(-x) + x(-cx + d)(-1)^n g(-x)
= (-ax + b)f_1(x) + x(-cx + d)g_1(x),
\]

since \( a < 0 \) and \( d \geq bc/a \), we get \( ad \leq bc \), and so \((-a)d \geq b(-c)\). By Corollary \[1.2\] \( F_1(x) \in \text{RZ} \) yielding \( F(x) \in \text{RZ} \).

The remaining possibility is when \( \alpha \) and \( \beta \) are negative.

If \( a > 0 \), by (c), we have \( r \leq 0 \) for each zero \( r \) of \( f \). Thus, all coefficients of \( f \) are nonpositive. Since \( g \) interlace \( f \), all coefficients of \( g \) are also nonpositive. Let

\[
f_2(x) := -f(x), \quad g_2(x) := -g(x).
\]
Thus, $f_2, g_2 \in PF$ and $g_2$ interlace $f_2$. From

$$-F(x) = (ax + b)(-f(x)) + x(cx + d)(-g(x)) = (ax + b)f_2(x) + x(cx + d)g_2(x),$$

since $a > 0$ and $d \geq bc/a$, Corollary 1.2 implies $-F(x) \in RZ$, and so $F(x) \in RZ$.

If $a < 0$, by (d) all zero $r$ of $f$ are nonnegative. Thus, the coefficients of $f$ and $g$ are alternating in sign. Since $\alpha$ is negative, we get $(-1)^{i} \alpha_{n-i} \leq 0$ ($0 \leq i \leq n$).

Let

$$f_3(x) := (-1)^{n+1}f(-x), \quad g_3(x) := (-1)^{n}g(-x), \quad F_3(x) := (-1)^{n+1}F(-x).$$

From

$$f_3(x) = (-1)^{n+1}f(-x) = (-1)^{n+1}[\alpha_n(-x)^n + \alpha_{n-1}(-x)^{n-1} + \cdots + \alpha_1(-x) + \alpha_0]$$

$$= (-1)\alpha_n x^n + (-1)^{n-1}2\alpha_{n-1} x^{n-1} + \cdots + (-1)^n \alpha_1 x + (-1)^{n+1} \alpha_0,$$

we see that all coefficients of $f_3(x)$ are nonnegative, i.e., $f_3 \in PF$. Similarly, $g_3 \in PF$ and $g_3$ interlaces $f_3$. Thus,

$$F_3(x) = (-1)^{n+1}[(-ax + b)f(-x) - x(-cx + d)g(-x)]$$

$$= (-ax + b)(-1)^{n+1}f(-x) + x(-cx + d)(-1)^n g(-x)$$

$$= (-ax + b)f_3(x) + x(-cx + d)g_3(x).$$

Since $a < 0$ and $d \geq bc/a$, we have $(-a)d \geq b(-c)$, and so $F_3 \in RZ$ showing that $F \in RZ$.

There remains to check the final two additional assertions.

If $c > 0$ and $-d/c \leq r \leq 0$, for each zero $r$ of $f$, then $r(c \tau + d) \leq 0$. If the leading coefficient of $f$ and $F$ have same sign, by Theorem 1.3, $f$ interlaces $F$. If the leading coefficient of $f$ and $F$ have different sign, then $a < 0$. From (d), each zero $r$ of $f$ is nonnegative. This implies that the zeros of $f$ can only be 0, and so $f(x) = \alpha x^n$, $g(x) = \beta x^{n-1}$. Thus,

$$F(x) = (ax + b)(\alpha x^n) + x(cx + d)(\beta x^{n-1}) = (\alpha \alpha + c\beta)x^{n+1} + (b\alpha + d\beta)x^n$$

$$= x^n[(\alpha \alpha + c\beta)x + (b\alpha + d\beta)]$$

showing that $f$ interlace $F$.

If $c = 0$ and $r \leq 0$ for each zero $r$ of $f$, we treat two cases separately.

Case $a < 0$. By (d), we have $r \geq 0$ for each zero $r$ of $f$. We must then have $r = 0$, and so $f(x) = \alpha x^n$ and $g(x) = \beta x^{n-1}$, by (a). From

$$F(x) = (ax + b)f(x) + (dx)g(x) = (ax + b)\alpha x^n + (dx)\beta x^{n-1}$$

$$= (\alpha ax + b\alpha + d\beta)x^n,$$
we conclude that $f$ interlaces $F$.

Case $a > 0$. By (b), the leading coefficients of $F$ and $g$ have same sign. Since $c = 0$ and $d \geq bc/a$, we get $d \geq 0$ and so $dr \leq 0$ for each zero $r$ of $f$. By (b), we get $\deg F = \deg f + 1$, and by Theorem 1.3, we conclude that $f$ interlace $F$. $\square$

We end this note with an example.

Example 2.2. Let $f(x) = -x^2 - 4x - 3$, $g(x) = -x - 2$, $a = 2$, $b = 3$, $c = 2$ and $d = 8$. We have $F(x) = -4x^3 - 23x^2 - 34x - 9$ such that all zeros of $F$ are $\approx -3.51235$, $\approx -1.9006$ and $\approx -0.33705$. Thus $F \in RZ$ and $f$ interlaces $F$.

References


