On two diophantine equations

\[ 4^x + 7^y = z^2 \text{ and } 4^x + 11^y = z^2 \]

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Abstract

In this paper, we show that diophantine equations \(4^x + 7^y = z^2\) and \(4^x + 11^y = z^2\) have no solution in non-negative integer.

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1 Introduction

In 2002, J. Sandor studied two diophantine equations \(3^x + 3^y = 6^z\) and \(4^x + 18^y = 22^z\). After that D. Acu (2007) studied the diophantine equation \(2^x + \)
5^y = z^2. He found that this equation has exactly two solutions in non-negative integer \((x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}\). Inspired by [1], we study two diophantine equations

\[4^x + 7^y = z^2 \quad \text{and} \quad 4^x + 11^y = z^2,\]

where \(x, y \) and \(z \) are non-negative integers.

## 2 Main Results

In this study, we use Catalan’s conjecture (see [4]). It is proved there that the only solution in integers \(a > 1, b > 1, x > 1 \) and \(y > 1 \) of the equation

\[a^x + b^y = 1\]

is \(a = y = 3 \) and \(b = x = 2 \). Now we have the followings.

**Theorem 2.1.** The diophantine equation \(4^x + 7^y = z^2\) has on solution in non-negative integers.

**Proof.** From the diophantine equation \(4^x + 7^y = z^2\), we consider in 3 cases.

**Case 1:** \(x = 0\). We have \(7^y = z^2 - 1 = (z - 1)(z + 1)\). Then there are non-negative integers \(a \) and \(b \) such that \(7^a = z - 1, 7^b = z + 1, a < b \) and \(a + b = y\). So we have \(7^a(7^b - a - 1) = 7^b - 7^a = (z + 1) - (z - 1) = 2\). Therefore, \(7^a = 1\) or \(a = 0\). It follows that \(z = 0\) and \(7^b = z + 1 = 3\). This is impossible.

**Case 2:** \(y = 0\). We have \(2^{2x} = 4^x = z^2 - 1 = (z - 1)(z + 1)\). Then there are non-negative integers \(a \) and \(b \) such that \(2^a = z - 1, 2^b = z + 1, a < b \) and \(a + b = 2x\). Therefore, \(2^a \cdot (2^b - a - 1) = 2^b - 2^a = (z + 1) - (z - 1) = 2\). It follows that \(2^a = 1\) or \(2^a = 2\). That is \(a = 0\) or \(a = 1\). If \(a = 0\) then \(z = 2\) and \(2^b = 3\). This is impossible. Thus \(a = 1\). This implies that \(z = 3\) and \(b = 2\). Thus \(2x = a + b = 3\). That is, \(x\) is not integer which is a contradiction.

**Case 3:** \(x > 0\) and \(y > 0\). We have \(7^y = z^2 - 4^x = (z - 2^x)(z + 2^x)\). Then there are non-negative integers \(a \) and \(b \) such that \(7^a = z - 2^x, 7^b = z + 2^x, a < b \) and \(a + b = y\). Therefore, \(7^a(7^b - a - 1) = 7^b - 7^a = (z + 2^x) - (z - 2^x) = 2^{x+1}\). It follows that \(7^a = 1\) and \(7^b - a - 2^{x+1} = 1\). By Catalan’s conjecture, we can conclude that this diophantine equation has no solution. The theorem is proved.

**Theorem 2.2.** The diophantine equation \(4^x + 11^y = z^2\) has on solution in non-negative integers.

**Proof.** From the diophantine equation \(4^x + 11^y = z^2\), we consider in 3 cases.

**Case 1:** \(x = 0\). We have \(11^y = z^2 - 1 = (z - 1)(z + 1)\). Then there are non-negative integers \(a \) and \(b \) such that \(11^a = z - 1, 11^b = z + 1, a < b \) and
On two diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$

$a + b = y$. So we have $11^a(11^{b-a} - 1) = 11^b - 11^a = (z + 1) - (z - 1) = 2$. Therefore, $11^a = 1$ or $a = 0$. It follows that $z = 0$ and $11^b = z + 1 = 3$. This is impossible.

Case 2: $y = 0$. We have $2^{2x} = 4^x = z^2 - 1 = (z - 1)(z + 1)$. Then there are non-negative integers $a$ and $b$ such that $2^a = z - 1$, $2^b = z + 1$, $a < b$ and $a + b = 2x$. Therefore, $2^a \cdot (2^{b-a} - 1) = 2^b - 2^a = (z + 1) - (z - 1) = 2$. It follows that $2^a = 1$ or $2^a = 2$. That is $a = 0$ or $a = 1$. If $a = 0$ then $z = 2$ and $2^b = 3$. This is impossible. Thus $a = 1$. This implies that $z = 3$ and $b = 2$. Thus $2x = a + b = 3$. That is, $x$ is not integer which is a contradiction.

Case 3: $x > 0$ and $y > 0$. We have $11^y = z^2 - 4^x = (z - 2^x)(z + 2^x)$. Then there are non-negative integers $a$ and $b$ such that $11^a = z - 2^x$, $11^b = z + 2^x$, $a < b$ and $a + b = y$. Therefore, $11^a(11^{b-a} - 1) = 11^b - 11^a = (z + 2^x) - (z - 2^x) = 2^{x+1}$. It follows that $11^a = 1$ and $11^{b-a} = 2^{x+1} = 1$. By Catalan’s conjecture, we can conclude that this diophantine equation has no solution. The theorem is proved.

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References


