Weak convergence theorem for finding common fixed points of a families of nonexpansive mappings and a nonspreading mapping in Hilbert spaces

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Abstract

In this paper, we introduce an iterative method and prove a weak convergence theorem for finding common fixed points of a families of nonexpansive mappings and a nonspreading mapping in Hilbert spaces. Moreover, we apply our result to finding common element of a solution set of equilibrium problem with a relaxed monotone mapping and a common fixed point set nonspreading mappings. Using the result, we improve and unify several results in fixed point problems and equilibrium problems.

Keywords: Equilibrium problem, Fixed point problem, Nonexpansive mapping, Nonspreading mapping.

1 Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$. A mapping $F$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle,$$

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for all $x, y \in C$; see, for instance, [2, 5, 6, 15, 17]. On the other hand, a mapping $Q : C \to C$ is said to be quasi-nonexpansive if $F(Q) \neq \emptyset$ and

$$
\|Qx - y\| \leq \|x - y\|,
$$

for all $x \in C$ and $y \in F(Q)$, where $F(Q)$ is the set of fixed points of $Q$. If $T : C \to C$ is nonexpansive and the set $F(T)$ of fixed points of $T$ is nonempty, then $T$ is quasi-nonexpansive.

Recently, Kohsaka and Takahashi [10] introduced the following nonlinear mapping: Let $E$ be a a Hilbert space and let $C$ be a nonempty closed convex subset of $E$. Then, a mapping $S : C \to C$ is said to be nonspreading if

$$
2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2,
$$

for all $x, y \in C$. We know in a Hilbert space that every firmly nonexpansive mapping is nonspreading and that if the set of fixed points of a nonspreading mapping is nonempty, the nonspreading mapping is quasi-nonexpansive; see [10]. Let $A : C \to H$ be a mapping of $C$ into $H$ is called monotone if

$$
\langle Au - Av, u - v \rangle \geq 0 \quad \forall u, v \in C.
$$

A mapping $A : C \to H$ is called $\lambda$-inverse-strongly monotone if there exists a positive real number $\lambda$ such that

$$
\langle Au - Av, u - v \rangle \geq \lambda \|Ax - Ay\|^2 \quad \forall u, v \in C.
$$

A mapping $T : C \to H$ is said to be relaxed $\eta - \alpha$ monotone if there exist a mapping $\eta : C \times C \to H$ and a function $\alpha : H \to \mathbb{R}$ positively homogeneous of degree $p$, that is, $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in H$ such that

$$
\langle Tx - Ty, \eta(x, y) \rangle \geq \alpha(x - y), \quad \forall x, y \in C,
$$

where $p > 1$ is a constant; see [4]. In the case of $\eta(x, y) = x - y$ for all $x, y \in C$, $T$ is said to be relaxed $\alpha$-monotone. In the case of $\eta(x, y) = x - y$ for all $x, y \in C$ and $\alpha(z) = k\|z\|^p$, where $p > 1$ and $k > 0$, $T$ is said to be $p$-monotone; see [7, 16, 21]. In fact, in this case, if $p = 2$, then $T$ is a $k$-strongly monotone mapping. Moreover, every monotone mapping is relaxed $\eta - \alpha$ monotone with $\eta(x, y) = x - y$ for all $x, y \in C$ and $\alpha = 0$.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for $F$ is to determine its equilibrium points, i.e. the set

$$
EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}.
$$
Many problems in physics, optimization, and economics require some elements of \( EP(F) \), see \(^2\) \(^3\) \(^{11}\) \(^{18}\) \(^{19}\) \(^{20}\). Several iterative methods have been proposed to solve the equilibrium problem, see for instance \(^3\) \(^{18}\) \(^{19}\) \(^{20}\). In 2005, Combettes and Hirstoaga \(^3\) introduced an iterative scheme for finding the best approximation to the initial data when \( EP(F) \) is nonempty and proved a strong convergence theorem.

The variational inequality problem is to find \( u \in C \) such that

\[
\langle Au, v - u \rangle \geq 0
\]

for all \( v \in C \). The set of solutions of the variational inequality is denoted by \( VI(C, A) \). The generalized equilibrium problem for \( F \) and \( A \) is to find \( x \in C \) such that

\[
F(x, y) + \langle Ax, y - x \rangle \geq 0 \quad \text{for all } y \in C.
\]

Problem (1.1) was introduced by Takahashi and Takahashi \(^{19}\) and the set of solution of (1.1) is denoted by \( GEP(F, A) \). The generalized mixed equilibrium problem for \( F, \psi \) and \( A \) is to find \( x \in C \) such that

\[
F(x, y) + \phi(y) - \phi(x) + \langle Ax, y - x \rangle \geq 0 \quad \text{for all } y \in C.
\]

(1.2)

Recently, Wang et al. \(^{22}\) introduce the generalized mixed equilibrium problem with a relaxed monotone mapping, that is, to find \( x \in C \) such that

\[
F(x, y) + \langle Tx, \eta(y, x) \rangle + \langle Ax, y - x \rangle \geq 0 \quad \text{for all } y \in C.
\]

(1.3)

The set of solution of (1.3) is denoted by \( GEP(F, T) \).

On the other hand, Halpern \(^{8}\) introduced the following iterative scheme for approximating a fixed point of \( T \):

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n
\]

(1.4)

for all \( n \in \mathbb{N} \), where \( x_1 = x \in C \) and \( \{\alpha_n\} \) is a sequence of \([0, 1]\). Recently, Aoyama et al. \(^{1}\) introduced a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings. Let \( x_1 = x \in C \) and

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)T_n x_n
\]

(1.5)

for all \( n \in \mathbb{N} \), where \( C \) is a nonempty closed convex subset of a Banach space, \( \{\alpha_n\} \) is a sequence in \([0, 1]\) and \( \{T_n\} \) is a sequence of nonexpansive mappings of \( C \) into itself which satisfies the AKTT-condition, that is,

\[
\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in C\} < \infty.
\]

(1.6)
They proved that the sequence \( \{x_n\} \) defined by (1.5) converges strongly to a common fixed point of \( \{T_n\} \).

In this paper, motivated by Plubtieng and Thammathiwat [14], Iemoto and Takahashi [9], Wang et al. [22], we introduce a new iterative sequence and prove a weak convergence theorem for finding common fixed points of a families of non-expansive mappings and a nonspreading mapping in Hilbert spaces. Moreover, we apply our result to finding common element of a solution set of equilibrium problem with a relaxed monotone mapping and a common fixed point set non-spreading mappings.

2 Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section. Let \( H \) be a real Hilbert space with inner product \( \langle ., . \rangle \) and norm \( \|\cdot\| \). In a Hilbert space, it is known that

\[
\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.
\]

By definition of the metric projection \( P_C \) we have known that \( P_C \) is a nonexpansive mapping of \( H \) onto \( C \) and satisfies

\[
\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y \rangle, \quad \forall x, y \in H.
\]

Further, for any \( x \in H \) and \( y \in C, \ y = P_Cx \) if and only if \( \langle x - y, y - z \rangle \geq 0, \ \forall z \in C \).

A space \( X \) is said to satisfy Opial’s condition [12] if for each sequence \( \{x_n\}_{n=1}^\infty \) in \( X \) which converges weakly to point \( x \in X \), we have

\[
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x
\]

and

\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x.
\]

Lemma 2.1. [10] Let \( H \) be a Hilbert space, \( C \) a nonempty closed convex subset of \( H \). Let \( S \) be a nonspreading mapping of \( C \) into itself. Then \( F(S) \) is closed and convex.

In order to prove the main result, we shall use the following lemmas in the sequel.

Lemma 2.2. [9] Let \( H \) be a Hilbert space, \( C \) a closed convex subset of \( H \), and \( S : C \to C \) a nonspreading mapping with \( F(S) \neq \emptyset \). Then \( S \) is semiclosed, i.e., \( x_n \rightrightarrows u \) and \( x_n - Sx_n \to 0 \) imply \( u \in F(S) \).
Lemma 2.3. Let $C$ be a nonempty bounded closed convex subset of Hilbert space $E$ and $\{T_n\}$ a sequence of mappings of $C$ into itself. Suppose that
\[
\lim_{k,l \to \infty} \rho_k^l = 0 \tag{2.1}
\]
where $\rho_k^l = \sup \{\|T_kz - T_lz\| : z \in C\} < \infty$, for all $k, l \in \mathbb{N}$. Then for each $x \in C$, $\{T_nx\}$ converges strongly to some point of $C$. Moreover, let $T$ be a mapping from $C$ into itself defined by
\[
Tx = \lim_{n \to \infty} T_nx, \quad \text{for all } x \in C.
\]
Then $\limsup_{n \to \infty} \{\|Tz - T_nz\| : z \in C\} = 0$.

In fact, Aoyama et al. [1] proved Lemma 2.3 in case the sequence $\{T_n\}$ satisfies the AKTT-condition. We note that if a sequence $\{T_n\}$ satisfies the AKTT-condition then $\{T_n\}$ satisfies the condition (2.1).

3 Weak convergence theorem

In this section, we prove a weak convergence theorem for finding common fixed points of a families of nonexpansive mappings and a nonspreading mapping in a Hilbert space.

Theorem 3.1. Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $S$ be a nonspreading mapping of $C$ into itself and let $\{T_n\}$ be the sequences of nonexpansive mappings of $C$ into itself such that $F := F(S) \cap \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ such that $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$. Let $\{x_n\}$ be a sequence defined by $x_0 = x \in C$ and
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)ST_n x_n, \quad n \geq 0. \tag{3.1}
\]
Suppose that $\{T_n\}$ satisfy the AKTT-condition, $T$ be the mapping of $C$ into itself defined by $Ty = \lim_{n \to \infty} T_ny$ for all $y \in C$ such that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ and suppose that for any $v \in F$,
\[
\|T_nx_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - T_nx_n\|^2 + c_n,
\]
where $\lim_{n \to \infty} c_n = 0$. Then $\{x_n\}$ converges weakly to $\hat{z} \in F$.

Proof. Take a point $v \in F$ and put $y_n = T_nx_n$. We shall show that the sequences $\{x_n\}$ is bounded. First, we note that
\[
\|Sy_n - v\| \leq \|y_n - v\| = \|T_nx_n - v\| \leq \|x_n - v\|
\]
and hence
\[
\|x_{n+1} - v\|^2 = \|\alpha_n x_n + (1 - \alpha_n) Sy_n - v\|^2 = \|\alpha_n(x_n - v) + (1 - \alpha_n) (Sy_n - v)\|^2 \\
\leq \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) \|Sy_n - v\|^2 - \alpha_n(1 - \alpha_n) \|Sy_n - x_n\|^2 \\
\leq \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) \|y_n - v\|^2 - \alpha_n(1 - \alpha_n) \|Sy_n - x_n\|^2 \\
= \|x_n - v\|^2 - \alpha_n(1 - \alpha_n) \|Sy_n - x_n\|^2 \\
\leq \|x_n - v\|^2.
\] (3.2)

Then \(\{\|x_{n+1} - v\|\}\) is a decreasing sequence and therefore \(\lim_{n \to \infty} \|x_n - v\|\) exists. This implies that \(\{x_n\}, \{T_n x_n\}, \{y_n\}\) and \(\{Sy_n\}\) are bounded. By our assumption, we have
\[
\|T_n x_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - T_n x_n\|^2 + c_n,
\]
where \(\lim_{n \to \infty} c_n = 0\). Thus, we note that
\[
\|x_{n+1} - v\|^2 = \|\alpha_n x_n + (1 - \alpha_n) Sy_n - v\|^2 = \|\alpha_n(x_n - v) + (1 - \alpha_n) (Sy_n - v)\|^2 \\
\leq \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) \|Sy_n - v\|^2 \\
\leq \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) \|y_n - v\|^2 \\
= \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) \|T_n x_n - v\|^2 \\
\leq \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) (\|x_n - v\|^2 - \|x_n - T_n x_n\|^2 + c_n)
\] (3.3)
and hence
\[
(1 - \alpha_n) \|x_n - T_n x_n\|^2 \leq \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) \|x_n - v\|^2 \\
+ (1 - \alpha_n) c_n - \|x_{n+1} - v\|^2 \\
= \|x_n - v\|^2 - \|x_{n+1} - v\|^2 + (1 - \alpha_n) c_n.
\]
Since \(0 < a \leq \alpha_n \leq b < 1\), \(c_n \to 0\) and \(\lim_{n \to \infty} \|x_n - v\|^2 = \lim_{n \to \infty} \|x_{n+1} - v\|^2\), it follows that
\[
\|x_n - T_n x_n\| = \|x_n - y_n\| \to 0.
\] (3.4)
Since \(\{y_n\}\) is bounded, there exists a subsequence \(\{y_{n_i}\}\) of \(\{y_n\}\) which converges weakly to \(\hat{z}\). Without loss of generality, we can assume that \(y_{n_i} \rightharpoonup \hat{z}\). By Lemma 2.2, we have \(\hat{z} \in F(S)\). Since \(\lim_{n \to \infty} \|x_n - y_n\| \to 0\) and \(y_{n_i} \rightharpoonup \hat{z}\), we get \(x_{n_i} \rightharpoonup \hat{z}\).
We shall show that $\hat{z} \in F(T)$. From $\|T_n x_n - x_n\| \to 0$ and the AKTT-condition, we have $\|T x_n - x_n\| \leq \|T x_n - T_n x_n\| + \|T_n x_n - x_n\| \to 0$. We next show that $\hat{z} \in F(T)$. Assume $\hat{z} \notin F(T)$. Since $x_n \rightharpoonup \hat{z}$ and $\hat{z} \neq T \hat{z}$. By the Opial’s condition, we have
\[
\liminf_{n \to \infty} \|x_n - \hat{z}\| < \liminf_{n \to \infty} \|x_n - T \hat{z}\|
\leq \liminf_{n \to \infty} \|x_n - T x_n\| + \|T x_n - T \hat{z}\|
\leq \liminf_{n \to \infty} \|x_n - \hat{z}\|.
\]
This is a contradiction. So, we get $\hat{z} \in F(T)$. Hence $\hat{z} \in F$. Let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $\hat{z}$. We may show that $\hat{z} = \check{z}$, suppose not. Since $\lim_{n \to \infty} \|x_n - v\|$ exists for all $v \in F$, it follows by the Opial’s condition that
\[
\lim_{n \to \infty} \|x_n - \check{z}\| = \liminf_{i \to \infty} \|x_{n_i} - \check{z}\| < \liminf_{i \to \infty} \|x_{n_i} - \hat{z}\| = \lim_{n \to \infty} \|x_n - \hat{z}\|
= \liminf_{i \to \infty} \|x_{n_k} - \check{z}\| < \liminf_{k \to \infty} \|x_{n_k} - \hat{z}\| = \lim_{n \to \infty} \|x_n - \hat{z}\|.
\]
This is a contradiction. Thus, we have $\hat{z} = \check{z}$. This implies that $\{x_n\}$ converges weakly to $\check{z} \in F$. This completes the proof.

\textbf{Corollary 3.2.} Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $S$ be a nonspreading mapping of $C$ into itself and let $\{T_n\}$ be the sequences of firmly nonexpansive mappings of $C$ into itself such that $F := F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$ is nonempty. Let $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ such that $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Let $\{x_n\}$ be a sequence defined by $x_0 = x \in C$ and
\begin{equation}
\label{eq:3.5}
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) ST_n x_n, \quad n \geq 0.
\end{equation}
Suppose that $\{T_n\}$ satisfy the AKTT-condition and $T$ be the mappings of $C$ into itself defined by $Ty = \lim_{n \to \infty} T_n y$ for all $y \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges weakly to $\check{z} \in F$.

\textbf{Proof.} Since $\{T_n\}$ is firmly nonexpansive, it follows that
\[
\|T_n x_n - v\|^2 = \|T_n x_n - T_n v\|^2
\leq \langle T_n x_n - v, x_n - v \rangle
= \frac{1}{2} (\|T_n x_n - v\|^2 + \|x_n - v\|^2 - \|x_n - T_n x_n\|^2),
\]
for all $v \in F$ and hence $\|T_n x_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - T_n x_n\|^2$. This implies that $\{T_n x_n\}$ satisfying condition in Theorem 3.1. So, we obtain the desired result by using Theorem 3.1. \qed
4 Applications

In this section, using Theorem 3.1, we prove weak convergence theorem for finding a common element of the set of solutions of equilibrium problem with a relaxed monotone mapping and the fixed point set of a nonspreading mapping in Hilbert space. Before, proving our theorems, we need the following lemmas.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that $F$ satisfies following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$.

(A2) $F$ is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$.

(A3) for each $x, y, z \in C$. \[ \lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y). \]

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

**Lemma 4.1.** [2] Let $C$ be a nonempty closed convex subset of $H$ and let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \text{ for all } y \in C.$$ 

The following lemma was also given in [3].

**Lemma 4.2.** [3] Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \}$$

for all $z \in H$. Then, the following hold:

(1) $T_r$ is single-valued;

(2) $T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$;

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.
Lemma 4.3. [22] Let $H$ be a real Hilbert space and let $C$ be a nonempty bounded closed convex subset of $H$. Let $T : C \rightarrow H$ be an $\eta$-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping and let $\Phi$ be a bifunction from $C \times C \rightarrow \mathbb{R}$ satisfying (A1), (A2), and (A4). Let $r > 0$ and define a mapping $T_r : H \rightarrow C$ as follows:

$$ T_r(x) = \{ z \in C : \Phi(z,y) + \langle Tz, \eta(y,z) \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \} $$

for all $x \in H$. Assume that

(i) $\eta(x,y) + \eta(y,x) = 0$, for all $x, y \in C$;

(ii) for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x,u) \rangle$ is convex and lower semicontinuous and the mapping $x \mapsto \langle Tu, \eta(v,x) \rangle$ is lower semicontinuous;

(iii) $\alpha : H \rightarrow \mathbb{R}$ is weakly lower semicontinuous;

(iv) for any $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$.

Then, the following holds:

(1) $T_r$ is single-valued;

(2) $T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$;

(3) $F(T_r) = EP(\Phi, T)$;

(4) $EP(\Phi, T)$ is closed and convex.

Theorem 4.4. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\psi : C \times C \rightarrow \mathbb{R}$ satisfying (A1)-(A4). Let $T$ be $\eta$-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping of $C$ into $H$ and let $S$ be a nonspreading mapping of $C$ into itself such that $F := F(S) \cap EP(\psi, T) \neq \emptyset$. Suppose $x_0 = x \in C$ and define the sequence $\{x_n\}$ and $\{y_n\}$ by

$$ \begin{cases} 
\psi(y_n, y) + \langle Ty_n, \eta(x, y_n) \rangle + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle \geq 0, \forall y \in C, \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sy_n,
\end{cases} $$

for all $n \in \mathbb{N}$, where $\{r_n\} \in (0, \infty)$ with $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ with $\lim inf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Then $\{x_n\}$ converges weakly to $\hat{z} \in F$.  

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Proof. Setting $T_n \equiv T_{r_n}$ in Theorem 3.1 and putting $y_n = T_{r_n}x_n$. Let $v \in F$ and let $B$ be a bounded subset of $C$. For $n \in \mathbb{N}$, let $z_n = T_{r_n}z$. we first prove that
\[
\sum_{n=1}^{\infty} \sup \left\{ \|T_{r_n+1}z - T_{r_n}z\| : z \in B \right\} < \infty \tag{4.1}
\]
for any bounded subset $B$ of $C$. We note that
\[
\psi(z_n, y) + \langle Tz_n, \eta(y, z_n) \rangle + \frac{1}{r_n} \langle y - z_n, z_n - z \rangle \geq 0 \tag{4.2}
\]
for all $y \in C$ and
\[
\psi(z_{n+1}, y) + \langle Tz_{n+1}, \eta(y, z_{n+1}) \rangle + \frac{1}{r_{n+1}} \langle y - z_{n+1}, z_{n+1} - z \rangle \geq 0 \tag{4.3}
\]
for all $y \in C$. Setting $y = z_{n+1}$ in (4.2) and $y = z_n$ in (4.3), we have
\[
\psi(z_n, z_{n+1}) + \langle Tz_n, \eta(z_{n+1}, z_n) \rangle + \frac{1}{r_n} \langle z_{n+1} - z_n, z_n - z \rangle \geq 0
\]
and
\[
\psi(z_{n+1}, z_n) + \langle Tz_{n+1}, \eta(z_{n+1}, z_n) \rangle + \frac{1}{r_{n+1}} \langle z_n - z_{n+1}, z_{n+1} - z \rangle \geq 0.
\]
Adding the two inequalities and by (A2), we have
\[
\langle Tz_n - Tz_{n+1}, \eta(z_{n+1}, z_n) \rangle + \left( \frac{z_{n+1} - z_n}{r_n} - \frac{z_{n+1} - z_n}{r_{n+1}} \right) \geq 0.
\]
Thus, we have
\[
\left( \frac{z_{n+1} - z_n}{r_n} - \frac{z_{n+1} - z_n}{r_{n+1}} \right) \geq \langle Tz_{n+1} - Tz_n, \eta(z_{n+1}, z_n) \rangle
\]
and hence
\[
\langle z_{n+1} - z_n, z_n - z_{n+1} \rangle + \left( \frac{z_{n+1} - z_n}{r_n} - \frac{r_n}{r_{n+1}} \right) (z_{n+1} - z) \geq \langle Tz_{n+1} - Tz_n, \eta(z_{n+1}, z_n) \rangle \tag{4.4}
\]
Since $T$ is relaxed $\eta - \alpha$ monotone mapping and $r > 0$, we have
\[
- \|z_{n+1} - z_n\|^2 + \left( \frac{z_{n+1} - z_n}{r_n} - \frac{r_n}{r_{n+1}} \right) (z_{n+1} - z) \geq r\alpha(z_{n+1} - z_n).
\tag{4.5}
Similarly, by exchanging the position of $z_{n+1}$ and $z_n$ in (4.4), we get

$$\langle z_n - z_{n+1}, z_{n+1} - z_n \rangle + \left( z_n - z_{n+1}, (1 - \frac{r_{n+1}}{r_n})(z_n - z) \right) \geq \langle Tz_n - Tz_{n+1}, \eta(z_n, z_{n+1}) \rangle. \tag{4.6}$$

and hence

$$-\|z_n - z_{n+1}\|^2 + \left( z_n - z_{n+1}, \left(1 - \frac{r_{n+1}}{r_n}\right)(z_n - z) \right) \geq r\alpha(z_n - z_{n+1}). \tag{4.7}$$

Adding (4.5) and (4.7), we have

$$2\|z_{n+1} - z_n\|^2 \leq \left(1 - \frac{r_n}{r_{n+1}}\right)\|z_{n+1} - z\| + \left(1 - \frac{r_{n+1}}{r_n}\right)\|z_n - z\|.$$

Thus $\|z_{n+1} - z_n\| \leq \frac{1}{2} \left[ \left|1 - \frac{r_n}{r_{n+1}}\right|\|z_{n+1} - z\| + \left|1 - \frac{r_{n+1}}{r_n}\right|\|z_n - z\| \right].$ Without loss of generality, let us assume that there exists a real number $b$ such that $r_n > b > 0$ for all $n \in \mathbb{N}$. Then

$$\|T_{r_{n+1}}z - T_{r_n}z\| = \|z_{n+1} - z_n\| \leq \frac{1}{2} \left[ \left|1 - \frac{r_n}{r_{n+1}}\right|\|T_{r_{n+1}}z - z\| + \left|1 - \frac{r_{n+1}}{r_n}\right|\|T_{r_n}z - z\| \right].$$

Let $u \in EP(\psi, T)$ and $M = \sup \{\|z - u\| : z \in B\}$. Then

$$\|T_{r_{n+1}}z - z\| \leq \|T_{r_{n+1}}z - u\| + \|u - z\| = \|T_{r_{n+1}}z - T_{r_{n+1}}u\| + \|u - z\| \leq 2\|z - u\|. $$
Similarly, we note that \( \|T_{r_n} z - z\| \leq 2\|z - u\| \). Thus, we have

\[
\|T_{r_{n+1}} z - T_{r_n} z\| \leq \frac{1}{2} \left[ \frac{1}{b} r_{n+1} - r_n \|T_{r_{n+1}} z - z\| + \frac{1}{b} r_n - r_{n+1} \|T_{r_n} z - z\| \right]
\leq \frac{1}{2} \left[ \frac{4M}{b} |r_{n+1} - r_n| \right]
= \frac{2M}{b} |r_{n+1} - r_n|.
\]

Hence \( \sup \{\|T_{r_{n+1}} z - T_{r_n} z\| : z \in B\} \leq \frac{2M}{b} |r_{n+1} - r_n| \). Since \( \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty \), we obtain \( \sum_{n=1}^{\infty} \sup \{\|T_{r_{n+1}} z - T_{r_n} z\| : z \in B\} < \infty \). By Lemma 2.3, we define a mapping \( T \) by \( Tx = \lim_{n \to \infty} T_{r_n} x \) for all \( x \in C \).

Next, we prove that \( F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n}) \). It easy to see that \( \bigcap_{n=1}^{\infty} F(T_{r_n}) \subset F(T) \). Let \( w \in F(T) \). For \( n \in \mathbb{N} \), let \( w_n = T_{r_n} w \). Then

\[
\psi(w_n, y) + \langle Tw_n, \eta(y, w_n) \rangle + \frac{1}{r_n} (y - w_n, w_n - w) \geq 0
\]

for all \( y \in C \). By (A2), we obtain \( \frac{1}{r_n} \langle y - w_n, w_n - w \rangle \geq \psi(y, w_n) + \langle Tw_n, \eta(w_n, y) \rangle \) for all \( y \in C \). Since \( w_n \to w \) and from (A4), we have \( 0 \geq \psi(y, w) + \langle Tw, \eta(w, y) \rangle \) for all \( y \in C \). Put \( u_t = ty + (1 - t)w \) for all \( t \in (0, 1] \) and \( y \in C \). Thus, we note that

\[
0 = \psi(u_t, u_t) + \langle Tw, \eta(u_t, u_t) \rangle + t \psi(ty + (1 - t)w, y) + (1 - t) \psi(t(1 - t)w, y, u_t) + t \langle Tw, \eta(y, u_t) \rangle
\]

So, \( \psi(ty + (1 - t)w, y) + \langle Tw, \eta(y, u_t) \rangle \geq 0 \) for all \( y \in C \). Letting \( t \to 0^+ \) and using (A3), we obtain \( \psi(w, y) + \langle Tw, \eta(y, w) \rangle \geq 0 \) for all \( y \in C \). Thus \( w \in EP(\psi, T) \). It follows that \( w \in \bigcap_{n=1}^{\infty} F(T_{r_n}) \) and it easy to see that

\[
\|T_{r_n} x_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - T_{r_n} x_n\|^2.
\]

Thus \( \{T_{r_n} x_n\} \) satisfying condition in Theorem 3.1. So, we obtain the desired result by using Theorem 3.1. 

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References


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